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# REPRESENTATIONS OF $SO(5)_q$ AND NON-MINIMAL q-DEFORMATION

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**Abstract**

Representations of $SO(5)_q$ can be constructed on bases such that either the Chevalley triplet ($e_1$, $f_1$, $h_1$) or ($e_2$, $f_2$, $h_2$) has the standard $SU(2)_q$ matrix elements. The other triplet in each cases has a more complicated action. The q-deformation of such representations present striking differences. In one case a **non-minimal** deformation is found to be essential. This is explained and illustrated below. Broader interests of a parallel use of the two bases are pointed out.

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The $q$-deformation of representations of non-simply laced Lie algebras (with roots of unequal length ) present special problems. This is illustrated by comparing, for particular cases, the respective $q$-deformations of irreducible representations of $SO(5)$ in two bases. Imposing the standard $SU(2)$ representations for the triplets of Chevalley generators associated to the shorter and the longer root of $SO(5)$ by turn lead to surprisingly different consequences concerning $q$-deformation. Irreducible representations of $SO(5)$ are characterized by two invariant parameters $n_1$ and $n_2$ ($n_1 \geq n_2$, both integer or half-integer). In this note we will consider only the cases

$$(\mathbf{1}) \qquad n_2 = 0, \quad \text{and}$$

$$(\mathbf{2}) \qquad n_2 = n_1$$

Upto now only for these two cases the solutions are complete. But even within such restrictions remarkable features arise. For $n_2 = n_1$ one encounters an example (defined below) of **non-minimal** $q$-deformation which is our main result here. The case $n_2 = 0$, needing essentially **minimal** deformation serves as a contrast. By **minimal** $q$-deformation we mean [1] introduction of $q$-brackets for each factor in the classical matrix elements of the Chevalley generators acting on a suitably parametrized set of basis states. **Non-minimal** means a departure from this involving subtler, more complicated $q$-deformations of some factors giving back again, of course, the same classical limit. The significances of these definitions will be more explicit after the examples to follow.

The Chevalley generators consist of two triplets $(e_1, \ f_1, \ h_1)$, $(e_2, \ f_2, \ h_2)$ corresponding to the roots 1 and 2 respectively. The standard Drinfeld-Jimbo construction for $SO(5)_q$ is, with commuting Cartan generators $q^{\pm h_1}$, $q^{\pm h_2}$,

$$q^{\pm h_1} e_1 = q^{\pm 1} e_1 q^{\pm h_1}, \quad q^{\pm h_1} f_1 = q^{\mp 1} f_1 q^{\pm h_1}$$

$$q^{\pm 2h_2} e_1 = q^{\mp 1} e_1 q^{\pm 2h_2}, \quad q^{\pm 2h_2} f_1 = q^{\pm 1} f_1 q^{\pm 2h_2}$$

$$q^{\pm h_1} e_2 = q^{\mp 1} e_2 q^{\pm h_1}, \quad q^{\pm h_1} f_2 = q^{\pm 1} f_2 q^{\pm h_1}$$

$$q^{\pm h_2} e_2 = q^{\pm 1} e_2 q^{\pm h_2}, \quad q^{\pm h_2} f_2 = q^{\mp 1} f_2 q^{\pm h_2}$$

$$[e_1, f_2] = 0, \quad [e_2, f_1] = 0$$

$$[e_1, f_1] = \ [2h_1] \equiv \left( \frac{q^{2h_1} - q^{-2h_1}}{q - q^{-1}} \right)$$

$$[e_2, f_2] = [2h_2]_2 \equiv \left( \frac{q^{4h_2} - q^{-4h_2}}{q^2 - q^{-2}} \right)$$

$$e_2 e_3^{(\pm)} = q^{\pm 2} e_3^{(\pm)} e_2 \tag{1}$$

$$f_3^{(\pm)} f_2 = q^{\pm 2} f_2 f_3^{(\pm)}$$

$$[e_1, e_4] = 0, \quad [f_1, f_4] = 0$$

where we have defined

$$e_3^{(\pm)} = q^{\pm 1} e_1 e_2 - q^{\mp 1} e_2 e_1,$$

$$f_3^{(\pm)} = q^{\pm 1} f_2 f_1 - q^{\mp 1} f_1 f_2,$$

$$e_4 = q^{-1} e_1 e_3^{(+)} - q \, e_3^{(+)} e_1 = q \, e_1 e_3^{(-)} - q^{-1} e_3^{(-)} e_1, \tag{2}$$

$$f_4 = q^{-1} f_3^{(+)} f_1 - q \, f_1 f_3^{(+)} = q \, f_3^{(-)} f_1 - q^{-1} f_1 f_3^{(-)}.$$

The coproducts, counits and antipodes are the standard ones.

For subsequent, convenient, use we define also

$$q^{\pm M} = q^{\pm h_1}, \quad q^{\pm(K-M)} = q^{\pm 2h_2}$$

$$q^{\pm M_2} = q^{\pm \frac{1}{2}(K-M)} = q^{\pm h_2} \tag{3}$$

$$q^{\pm M_4} = q^{\pm \frac{1}{2}(K+M)} = q^{\pm(h_1+h_2)}$$

The second order Casimir operator is [1]

$$A = \frac{1}{[2]} \left\{ \left( f_1 e_1 + [M][M+1] \right) \frac{[2K+3]_2}{[2K+3]} + [K][K+3] \right\}$$

$$+ \left( f_2 e_2 + \frac{1}{[2]^2} f_4 e_4 \right) + \frac{1}{[2]^2} \left( f_3^{(+)} e_3^{(+)} q^{2M+1} + f_3^{(-)} e_3^{(-)} q^{-2M-1} \right). \tag{4}$$

Though we will need in the following only the restricted cases mentioned before we state here the general result that on the space of states spanning the irreducible representation $(n_1, n_2)$

$$A = \frac{1}{[2]} \left\{ [n_1][n_1+3] + [n_2][n_2+1] \frac{[2n_1+3]_2}{[2n_1+3]} \right\} \mathbf{1}. \tag{5}$$

where $\mathbf{1}$ is the identity. (For $n_2 = 0$, $\frac{1}{2}$, $n_1$ this reduces to the results in[1].)

Our aim is to compare two bases for irreducible representations $(n_1, n_2)$ defined as follows.

**BASIS (1):** let

$$q^{\pm M}|j\ m\ k\ l> = q^{\pm m}|j\ m\ k\ l>$$

$$q^{\pm K}|j\ m\ k\ l> = q^{\pm k}|j\ m\ k\ l>$$

$$e_1|j\ m\ k\ l> = ([j-m]\ [j+m+1])^{1/2}|j\ m+1\ k\ l>$$

$$e_2|j\ m\ k\ l> =$$

$$([j-m+1][j-m+2])^{1/2}\sum_{l'} a(j,k,l,l')|j+1\ m-1\ k+1\ l'> \qquad (6)$$

$$+\ (\ [j+m][j+m-1])^{1/2}\sum_{l'} b(j,k,l,l')|j-1\ m-1\ k+1\ l'>$$

$$+\ (\ [j+m][j-m+1])^{1/2}\sum_{l'} c(j,k,l,l')|j\ m-1\ k+1\ l'>$$

We will consider (for generic $q$) only **real** matrix elements, when for any two states $|x>$, $|y>$,

$$< x|f_i|y> = < y|e_i|x> \qquad\qquad (i=1,\ 2)$$

The domains of the indices have been obtained. The patterns of multiplicities are subtle. They are presented below without the derivations.

(i) For $(n_1,\ n_2)$ integers

$$j = 0,\ 1,\cdots,\ n_1-1,\ n_1$$

$$m = -j,\ -j+1,\cdots,\ j-1,\ j$$

$$k = -l,\ -l+2,\cdots,\ l-2,\ l$$

$$l = 0,\ 1,\ 2\ \cdots \qquad\qquad (7)$$

$$j+l = n_1-n_2,\ n_1-n_2+1,\cdots,\ n_1+n_2$$

$$j-l-\frac{1}{2}\big(1-(-1)^{n_1+n_2-j-l}\big) = -n_1+n_2,\ -n_1+n_2+2,\cdots,\ n_1-n_2$$

(for the comparing with (2.14) of [1] note that when $n_1 = n_2$, $l = j$, $j-1$ for $j > 0$ and $l = 0$ for $j = 0$.)

(ii) For $(n_1,\ n_2)$ half integers

$$j = \frac{1}{2},\ \frac{3}{2}, \cdots,\ n_1 - 1,\ n_1$$

$$m = -j,\ -j+1, \cdots,\ j-1,\ j$$

$$k = -l,\ -l+1, \cdots,\ l-1,\ l$$

$$l = \frac{1}{2},\ \frac{3}{2}, \cdots$$

$$j + l = n_1 - n_2 + 1,\ n_1 - n_2 + 3, \cdots,\ n_1 + n_2$$

$$j - l = -n_1 + n_2,\ -n_1 + n_2 + 2, \cdots,\ n_1 - n_2.$$

(8)

Upto now the solutions for the reduced matrix elements $a$, $b$, $c$ satisfying all the necessary algebraic constraints have been obtained [1] for the cases

$$n_2 = 0,\ \frac{1}{2},\ n_1$$

when there is no multiplicity due to $l$ and one can consider states labelled $|j\ m\ k >$. For comparison with the case to follow we reproduce here, briefly, the results for $n_2 = 0$ and $n_1 = n_2$ (for $n_2 = \frac{1}{2}$, see [1]).

To **start with consider only generic** $q$ (real, positive). For $n_2 = 0$, $n_1 = n$;

$$a(j, k) = (q + q^{-1})^{-1} \left( \frac{[n - j - k]\ [n + j + k + 3]}{[2j + 1]\ [2j + 3]} \right)^{1/2}$$

$$b(j, k) = (q + q^{-1})^{-1} \left( \frac{[n + j - k + 1]\ [n - j + k + 2]}{[2j - 1]\ [2j + 1]} \right)^{1/2}$$

(9)

$$c(j, k) = 0$$

where

$$j = 0, 1, 2, \cdots, n$$

$$k = n - j, n - j - 2, \cdots, -(n - j - 2), -(n - j)$$

$$m = j, j - 1, \cdots, -(j - 1), -j.$$

For $n_2 = n_1 = n$ (integer or half integer)

$$a(j,k) = (q+q^{-1})^{-1} \left( \frac{[n-j]_2 \, [n+j+2]_2 \, [j+k+1] \, [j+k+2]}{[2j+3] \, [2j+1] \, [j+1]_2^2} \right)^{1/2}$$

$$b(j,k) = (q+q^{-1})^{-1} \left( \frac{[n-j+1]_2 \, [n+j+1]_2 \, [j-k] \, [j-k-1]}{[2j+1] \, [2j-1] \, [j]_2^2} \right)^{1/2} \qquad (10)$$

$$c(j,k) = (q+q^{-1})^{-1}[n+1]_2 \frac{([j-k] \, [j+k+1])^{1/2}}{[j+1]_2 \, [j]_2}$$

where

$$j = n, n-1, \cdots, 0(1/2)$$

$$k = j, j-1, \cdots, -(j-1), -j$$

$$m = j, j-1, \cdots, -(j-1), -j.$$

Apart from the limiting values of $n_2$ mentioned (the lowest 0 and $\frac{1}{2}$ and the highest $n_1$) not even the **classical** representations have yet been obtained for this basis. (see the detailed discussion and comparison of the situation with that in the Gelfand-Zetlin basis [2] given in [1].) But for the $n_2$ values mentioned above setting $q = 1$ and comparing with the generic $q$-case one sees essentially an example of **minimal** $q$-deformation. The only effect of unequal roots is the appearence of $[x]_2$ brackets along with $[x]$'s.

**BASIS (2):** Consider now the following basis states ($\varepsilon = \pm 1$, $\varepsilon' = \pm 1$)

$$q^{\pm M_2} | j_2 \, m_2 \, j_4 \, m_4 >= q^{\pm m_2} | j_2 \, m_2 \, j_4 \, m_4 >$$

$$q^{\pm M_4} | j_2 \, m_2 \, j_4 \, m_4 >= q^{\pm m_4} | j_2 \, m_2 \, j_4 \, m_4 >$$

$$e_2 | j_2 \, m_2 \, j_4 \, m_4 >= ([j_2 - m_2]_2 [j_2 + m_2 + 1]_2)^{1/2} | j_2 \, m_2 + 1 \, j_4 \, m_4 >$$

$$e_1 | j_2 \, m_2 \, j_4 \, m_4 >= \sum_{\epsilon, \, \epsilon'} \; ([j_2 - \epsilon \, m_2 + \frac{1+\epsilon}{2}]_2)^{1/2} \times \qquad (11)$$

$$c_{(\epsilon, \epsilon')}(j_2, j_4, m_4) \, | j_2 + \frac{\epsilon}{2} \; m_2 - \frac{1}{2} \; j_4 + \frac{\epsilon'}{2} \; m_4 + \frac{1}{2} >$$

with, again,

$$< x | f_i | y >=< y | e_i | x > \qquad (i = 1, \, 2)$$

The domain of the indices (again for generic $q$ ) are

$$j_2 = 0, \; \frac{1}{2}, \; 1, \cdots, \; \frac{n_1 + n_2}{2}$$

$$j_4 = 0, \; \frac{1}{2}, \; 1, \cdots, \; \frac{n_1 + n_2}{2}$$

$$m_2 = -j_2, \; -j_2 + 1, \cdots, \; j_2 - 1, \; j_2$$

$$m_4 = -j_4, \; -j_4 + 1, \cdots, \; j_4 - 1, \; j_4$$

such that

$$j_2 + j_4 = n_2, \; n_2 + 1, \cdots, \; n_1$$

$$j_2 - j_4 = -n_2, \; -n_2 + 1, \cdots, \; n_2 \tag{12}$$

Now for $q = 1$ a complete solution for the reduced elements $c_{\epsilon,\epsilon'}$ is available. This is the representation of Hughes [3]. Though it does not seem to be explicitly noted in the paper, the **shift operators** of [3] correspond directly to the Chevalley generators $(e_1, \; f_1)$. The solutions can be written, in our notations, as

$$c_{(\epsilon,\epsilon')}(j_2, j_4, m_4) = (j_4 + \epsilon' \, m_4 + \frac{1 + \epsilon'}{2})^{1/2} c_{(\epsilon,\epsilon')}(j_2, j_4) \quad (\epsilon, \epsilon' = \pm 1) \tag{13}$$

with

$$c_{(++)}(j_2, \; j_4) = c_{(--)}(j_2 + \frac{1}{2}, \; j_4 + \frac{1}{2})$$

$$= \left( \frac{(n_1 + j_2 + j_4 + 3)(n_1 - j_2 - j_4)(j_2 + j_4 + n_2 + 2)(j_2 + j_4 - n_2 + 1)}{(2j_2 + 1) \, (2j_2 + 2) \, (2j_4 + 1) \, (2j_4 + 2)} \right)^{1/2} \tag{14}$$

and

$$c_{(+-)}(j_2, \; j_4) = -c_{(-+)}(j_2 + \frac{1}{2}, \; j_4 - \frac{1}{2})$$

$$= \left( \frac{(n_1 + j_2 - j_4 + 2)(n_1 - j_2 + j_4 + 1)(j_2 - j_4 + n_2 + 1)(j_4 - j_2 + n_2)}{(2j_2 + 1) \, (2j_2 + 2) \, (2j_4) \, (2j_4 + 1)} \right)^{1/2} . \tag{15}$$

But now the $q$-deformation is the problem. As yet solutions have been obtained for the following two cases.

(i) For $n_2 = 0, \; n_1 = n$

$$j_4 = j_2 = 0, \; \frac{1}{2}, \; \cdots, \; \frac{n}{2}$$

$$c_{(+-)}(j_2, j_4, m_4) = c_{(-+)}(j_2, j_4, m_4) = 0$$

and

$$c_{(++)}(j_2,\ j_4,\ m_4) = ([j_4 +\ m_4 + 1]_2)^{1/2}\ c_{(++)}(j_2,\ j_4)$$

$$c_{(--)}(j_2,\ j_4,\ m_4) = ([j_4 -\ m_4]_2)^{1/2}\ c_{(--)}(j_2,\ j_4)$$

where

$$c_{(++)}(j_2,\ j_4) = c_{(--)}(j_2 + \frac{1}{2},\ j_4 + \frac{1}{2})$$
$$= \left( \frac{[n_1 + 2\,j_2 + 3][n_1 - 2\,j_2]}{[2j_2 + 1]_2[2j_2 + 2]_2} \right)^{1/2} \tag{16}$$

this is straightforward. The factorisation of $m_4$-dependance is what one would expect. One has a **minimal** $q$-deformation (with $q^2$-brackets appearing as well).

(ii) For $n_1 = n_2 = n$

$$j_2 + j_4 = n,\ \ j_2 = 0,\ \frac{1}{2}, \cdots,\ n$$

$$c_{(++)}(j_2,\ j_4,\ m_4) =\ c_{(--)}(j_2,\ j_4,\ m_4) = 0. \tag{17}$$

If one tries to impose for $c_{(\pm,\mp)}$ an $m_4$-dependance of the type one expects from the classical expression and the typical minimal deformation (found for $n_2 = 0$ say) one runs into a contradiction. The following remarkable solution has been found. One obtains,

$$c_{(+-)}(j_2,\ j_4,\ m_4) = ([n + 1]_2 - [j_2 +\ m_4 + 1]_2)^{1/2}\ c_{(+-)}(j_2,\ j_4)$$
$$c_{(-+)}(j_2,\ j_4,\ m_4) = ([n + 1]_2 - [j_2 -\ m_4]_2)^{1/2}\ c_{(-+)}(j_2,\ j_4) \tag{18}$$

where

$$c_{(+-)}(j_2) = -c_{(-+)}(j_2 + \frac{1}{2}) = \left( \frac{[2j_2 + 1]\,[2j_2 + 2]}{[2j_2 + 1]_2\,[2j_2 + 2]_2} \right)^{1/2} \tag{19}$$

One preserves the correct classical limit. But the $m_4$-dependance involves a strikingly non-minimal $q$-deformation prescription. (This, to our knowledge, is the first example of this kind.) One can express the square root of the difference of two brackets (appearing through $m_4$-dependance) as a square root of products of brackets through the identity

$$[x]_2 - [y]_2 = [x - y]\frac{[x + y]_2}{[x + y]}.$$

But now $m_4$ appears in the denominator on the right which is again quite unusual.

From the definition of $(e_4, f_4)$ one now obtains ( with $j_4 = n - j_2$ )

$$e_4|j_2\ m_2\ j_4\ m_4 >= -(q + q^{-1})\ \{([n+1]_2 - [j_2 - m_4]_2) \times$$

$$([n+1]_2 - [j_2 + m_4 + 1]_2)\}^{1/2}|j_2\ m_2\ j_4\ m_4 + 1 >$$

$$f_4|j_2\ m_2\ j_4\ m_4 >= -(q + q^{-1})\ \{([n+1]_2 - [j_2 - m_4 + 1]_2) \times$$

$$([n+1]_2 - [j_2 + m_4]_2)\}^{1/2}|j_2\ m_2\ j_4\ m_4 - 1 >$$

$$(20)$$

(the negative sign arises due to phase conventions.)

For comparison we note that for $n_2 = 0$ $(j_2 = j_4)$ one has

$$e_4|j_2\ m_2\ j_4\ m_4 >$$

$$= (q + q^{-1})\{[j_4 - m_4]_2[j_4 + m_4 + 1]_2\}^{1/2}|j_2\ m_2\ j_4\ m_4 + 1 >$$

$$f_4|j_2\ m_2\ j_4\ m_4 >$$

$$= (q + q^{-1})\{[j_4 + m_4]_2[j_4 - m_4 + 1]_2\}^{1/2}|j_2\ m_2\ j_4\ m_4 - 1 >$$

$$(21)$$

here the classical limit and the $SU(2)_q$ structure associated with $(\frac{e_4}{[2]}, \frac{f_4}{[2]}, q^{\pm M_4})$ are evident. For $n_2 = n_1$, the commutator $[e_4, f_4]$ is more complicated but, of course, has the same classical limit.

Studying the bases parallely has other interests than providing interesting exercises in $q$-deformation. We briefly mention two important aspect to be explored elsewhere.

(a) Suitably adapting familiar continuation techniques $SO(3, 2)_q$ and $SO(4, 1)_q$ representations can be obtained from basis (1) and basis (2) respectively.

(b) Under suitable contraction procedures again $q$-deformation of representations of different inhomogeneous algebras are obtained in the two cases. The contractions of basis (1) are discussed in [1]. Contracted representations arising from basis (2) will be presented elsewhere. Here possibilities of applications are particularly interesting.

The major remaining task is the explicit construction of $SO(5)_q$ representations for arbitrary admissible, $(n_1, n_2)$. The elegant formalism of Fiore [4] gives the deformations of only the vector representations of $SO(N)$. If one intends to cover the full range of invariants and indices some essential,

hard problems are encounted already at the level of $SO(5)_q$. Overcoming them is the motivation behind our efforts.

The basis (1) classical representations seem (so far) to permit relatively simple (minimal) $q$-deformation. But the intricate multiplicity patterns (presented here for the first time) indicate the difficulties of a ( even classical) general solution. The unsuitability of the classical Gelfand-Zetlin representations [2] for $q$-deformation was explained in [1]. The classical representations of Hughes [3] (starting point of our basis (2)) have attractive properties but their $q$-deformation presents unxpected problems. We hope to present a general solution for basis(2) in a following paper.

The domains of the indices were considered above for generic $q$. For $q$ a root of unity the situation ( concerning dimensions and the center) changes radically. Neverthless, the periodic and partially periodic irreducible representations for $q$ a root of unity can be obtained from generic $q$ ones using our formalism of **fractional parts** [5] [1]. This will not be discussed here. We refer, however, to section IV of [1] for explanations and references.

**ACKNOWLEDEGMENT:**

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5. The formalism of fractional parts for $q$ a root of unity was first presented in

   D. Arnaudon and A. Chakrabarti. Periodic and partially periodic representations of $SU(N)_q$., Comm. Math. Phys. 139, 461 (1991)

   Ref.1 cites more sources. A different approach (with supplementary references) can be found in

   D. Arnaudon and A. Chakrabarti, "Periodic Representations of $SO(5)_q$", Phys.lett. B 262, 68 (1991).